3.2 Effective Interest Rates and Annuities

Effective Interest Rates

The effective rate is the simple interest rate that is equivalent to what the compound interest rate would pay out over a given period of time.

Comparing simple and compound interest rates for a one year period will produce the formula for the effective rate.

Simple interest is $A = P(1 + r \cdot t)$; after one year $A = P(1 + r)$.

Compound interest is $A = P\left(1 + \frac{r}{n}\right)^{n \cdot t}$; after one year $A = P\left(1 + \frac{r}{n}\right)^{n}$.

Let E be the effective rate. Therefore $P(1 + E) = P\left(1 + \frac{r}{n}\right)^{n} \rightarrow 1 + E = \left(1 + \frac{r}{n}\right)^{n} \rightarrow E = \left(1 + \frac{r}{n}\right)^{n} - 1$

Effective Interest Rate

$$E = \left(1 + \frac{r}{n}\right)^{n} - 1$$

Where: $E =$ effective rate
$n =$ number of periods per year the interest is calculated
$r =$ interest rate per year

Example 1 Find the effective interest rate when the stated rate is 6% compounded quarterly.

► *Solution*: $r = 0.06,\ n = 4$

$$E = \left(1 + \frac{r}{n}\right)^{n} - 1 = \left(1 + \frac{0.06}{4}\right)^{4} - 1 = 0.06136 = 6.14\%$$

The effective rate is 6.14%. Note, the 6% is referred to as the **nominal rate**.

Example 2 Which savings account is a better investment: 5.25% compounded daily, or 5.3% compounded semi-annually?

► *Solution*: 5.25%: $r = 0.0525,\ n = 365$

$$E = \left(1 + \frac{r}{n}\right)^{n} - 1 = \left(1 + \frac{0.0525}{365}\right)^{365} - 1 = 0.0539 = 5.39\%$$

5.3%: $r = 0.053,\ n = 2$

$$E = \left(1 + \frac{r}{n}\right)^{n} - 1 = \left(1 + \frac{0.053}{2}\right)^{2} - 1 = 0.0537 = 5.37\%$$

The 5.25% daily rate is slightly better. This is an important consideration for a savings account over a long period of time.

Annuities

An annuity is a savings plan in which the investor makes a regular fixed payment into a compound interest account and the interest rate does not change for the length of the investment. This type of investment differs from compound interest accounts where the entire amount is invested at the beginning of the investment period. Annuities break up an investment into smaller payments; they are used by individuals for saving for things such as: house payments, college expenses, vacations, cars, or retirement.

Example 3 Find the future amount of an annuity where a \$1000 payment (investment) is made annually for 4 years at 5% compound interest.

►*Solution*: The payment is made at the end of each year.

End of first year: \$1000.00 payment

End of second year: $\$1000 \times 0.05 = \$50.00$ interest
\$1000.00 principal
\$1000.00 payment
Total: \$2050.00

End of third year: $\$2050.00 \times 0.05 = \$102.50$
\$2050.00 principal
\$1000.00 payment
Total: \$3152.50

End of fourth year: $\$3152.50 \times 0.05 = \$157.63$ interest
\$3152.50 principal
\$1000.00 payment
Total: \$4310.13

The future value of the annuity at the end of 4 years is \$4310.13

Calculating future amount through this method is very time consuming. Using the annuity formula is a better way.

The Annuity Formula

Before developing an annuity formula, a geometric series formula must be developed. A geometric series is a series where each term, except the first, is found by multiplying the preceding term by a constant, then finding the sum.

For example: $1 + 2 + 4 + 8 + 16$, and $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$ are two examples of geometric series.

<u>Developing the Geometric Series Formula</u>

Let $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1}$. Now multiply each side of the equation by r.

$r \cdot S_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n$. Next subtract the first equation from the second and solve for S_n.

$r \cdot S_n - S_n = ar^n - a$. Factor the equation.

$S_n(r-1) = a(r^n - 1)$. Divide both sides of the equation by $(r-1)$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

<u>Developing the Annuity Formula</u>

Let $R =$ regular investment amount, $t =$ time in years, $r =$ investment rate in years

Step 1: Start at 0, after one year the investment is R

Step 2: After two years, R is compounded once and there is an additional investment of R. $R(1+r) + R$

Step 3: After three years, the first R has been compounded twice, the second R has been compounded once, and there is an additional investment of R. $R(1+r)^2 + R(1+r) + R$

Step 4: Let $x = (1+r)$, the pattern for the sum F is: $F = Rx^{n-1} + Rx^{n-2} + \cdots + Rx^1 + R$
$= R(x^{n-1} + x^{n-2} + \cdots + x + 1)$

Step 5: Use the geometric series $S_n = \dfrac{a(r^n - 1)}{r - 1}$ for $x^{n-1} + x^{n-2} + \cdots + x + 1 \;\rightarrow\; \dfrac{x^n - 1}{x - 1}$

Step 6: Substitute $(1+r)$ for $x \;\rightarrow\; F = \dfrac{R\left[(1+r)^n - 1\right]}{r}$

Interest and payments can be compounded more than once per year. As with the compound interest formula, divide r by the number of compounding periods per year, n. The number of times interest is compounded in t years is $n \cdot t$.

Future Amount of an Annuity

$$F = \frac{R\left[\left(1 + \frac{r}{n}\right)^{n \cdot t} - 1\right]}{\frac{r}{n}}$$

Where: F is the future amount of the annuity; R is the regular periodic payment; r is the annual interest rate; n is the number of payments per year; t is the term of the annuity in years

Example 4 Find the future amount of an annuity with quarterly payments of \$600 at 5% compounded quarterly for 10 years.

► *Solution*: $R = \$600,\ r = 0.05,\ n = 4$, and $t = 10$

$$F = \frac{R\left[\left(1+\frac{r}{n}\right)^{n\cdot t}-1\right]}{\frac{r}{n}} = \frac{\$600\left[\left(1+\frac{0.05}{4}\right)^{4\cdot 10}-1\right]}{\frac{0.05}{4}} = \$30\,893.73$$

The future value of the annuity is \$30 893.73

Example 5 Suppose you want to save up to purchase a new car in 5 years. What amount must you save semi-monthly if the car will cost \$50 000 and interest is calculated semi-monthly at 6%?

► *Solution*: $F = \$50\,000,\ r = 0.06,\ n = 24$, and $t = 5$ years

$$F = \frac{R\left[\left(1+\frac{r}{n}\right)^{n\cdot t}-1\right]}{\frac{r}{n}} \rightarrow \$50\,000 = \frac{R\left[\left(1+\frac{0.06}{24}\right)^{24\cdot 5}-1\right]}{\frac{0.06}{24}} \rightarrow R = \frac{\$50\,000\left(\frac{0.06}{24}\right)}{\left[\left(1+\frac{0.06}{24}\right)^{24\cdot 5}-1\right]}$$

$$= \$357.80$$

You would have to save \$357.80 semi-monthly for 5 years to have a future value of \$50 000.

Example 6 What is the better investment: saving \$200 monthly at 4% compounded monthly for 8 years, or saving \$100 semi-monthly at 4% compounded semi-monthly for 8 years?

► *Solution*: Compounded monthly

$R = \$200,\ r = 0.04,\ n = 12$, and $t = 8$

$$F = \frac{R\left[\left(1+\frac{r}{n}\right)^{n\cdot t}-1\right]}{\frac{r}{n}} = \frac{\$200\left[\left(1+\frac{0.04}{12}\right)^{12\cdot 8}-1\right]}{\frac{0.04}{12}} = \$22\,583.71$$

Compounded semi-monthly

$R = \$100,\ r = 0.04,\ n = 24$, and $t = 8$

$$F = \frac{R\left[\left(1+\frac{r}{n}\right)^{n\cdot t}-1\right]}{\frac{r}{n}} = \frac{\$100\left[\left(1+\frac{0.04}{24}\right)^{24\cdot 8}-1\right]}{\frac{0.04}{24}} = \$22\,605.66$$

If interest is calculated more times per year, the final value is greater. Compounding semi-monthly returns \$21.95 more over 8 years.