8.6 Law of Cosines

There are four cases in which it is possible to solve a general triangle ABC. The Law of Sines is used for two of the cases (ASA or AAS), the Law of Cosines is used for the remaining cases (SAS and SSS).

The Law of Cosines

For any triangle ABC and corresponding sides a, b, and c:

$a^2 = b^2 + c^2 - 2bc\cos A$ $\qquad \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

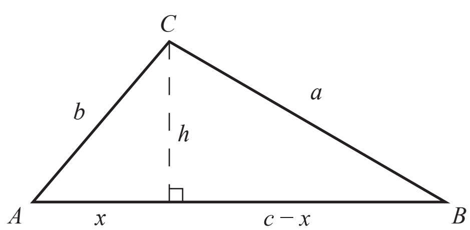
$b^2 = a^2 + c^2 - 2ac\cos B$ $\qquad \cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$c^2 = a^2 + b^2 - 2ab\cos C$ $\qquad \cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Note: If $\angle A = 90°,\ \cos A = 0 \ \rightarrow\ a^2 = b^2 + c^2$ *which is Pythagoras' Theorem*

Derivation of the Law of Cosines

Consider the oblique triangle ABC.



Length c is divided into two parts, x and $c - x$.

$$\cos A = \frac{x}{b} \ \rightarrow\ x = b\cos A$$

By the Pythagorean theorem: $b^2 = h^2 + x^2 \ \rightarrow\ h^2 = b^2 - x^2$

$$a^2 = h^2 + (c-x)^2 \ \rightarrow\ h^2 = a^2 - (c-x)^2$$

Equating these values for h^2: $a^2 - (c-x)^2 = b^2 - x^2$

$$\begin{aligned} a^2 &= b^2 - x^2 + (c-x)^2 \\ a^2 &= b^2 - x^2 + c^2 - 2cx + x^2 \\ a^2 &= b^2 + c^2 - 2cx \\ a^2 &= b^2 + c^2 - 2bc\cos A \quad \leftarrow b\cos A \textit{ is substituted for } x \end{aligned}$$

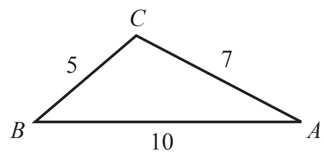
Just as easily, an altitude can be drawn from angle B to side b, and from angle A to side a, to show the remaining two cases of the Law of Cosines.

Using the Law of Cosines for SSS

Always find the largest angle first in a SSS problem. This will guarantee the other two angles are acute. The Law of Cosines never has the ambiguous case since a unique angle is always obtained between 0° and 180°.

Example 2 Solve ΔABC, given $a = 5,\ b = 7,\ c = 10$.

► *Solution*:



$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ 10^2 &= 5^2 + 7^2 - 2\cdot 5\cdot 7\cos C \\ 100 &= 25 + 49 - 70\cos C \\ 70\cos C &= 25 + 49 - 100 \\ 70\cos C &= -26 \\ \cos C &= -\frac{26}{70} \\ \angle C &= \cos^{-1}\left(-\frac{26}{70}\right) \\ \angle C &= 111.8^\circ \end{aligned}$$

The Law of Sines or the Law of Cosines can be used to find $\angle A$ or $\angle B$. The Law of Sines is easier to use.

$$\frac{\sin A}{a} = \frac{\sin C}{c} \rightarrow \frac{\sin A}{5} = \frac{\sin 111.8^\circ}{10} \rightarrow \sin A = \frac{5\sin 111.8^\circ}{10} \rightarrow \angle A = 27.7^\circ$$

Therefore $27.7^\circ + \angle B + 111.8^\circ = 180^\circ \rightarrow \angle B = 40.5^\circ$

*Note: If $\angle A$ was solved for first, it would lead to a **wrong** angle. Using the Law of Cosines, $\angle A$ would still be 27.7° but:*

$$\frac{\sin C}{c} = \frac{\sin A}{a} \rightarrow \frac{\sin C}{10} = \frac{\sin 27.7^\circ}{5} \rightarrow \sin C = \frac{10\sin 27.7^\circ}{5} \rightarrow \angle C = 68.2^\circ$$

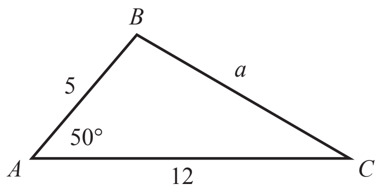
The correct result for $\angle C$ is $180^\circ - 68.2^\circ = 111.8^\circ$. To avoid this problem, always find the largest angle first in a SSS problem.

The only case in which a SSS triangle has no solution is when the two shorter sides of a triangle are less than the longest side. For example, a triangle cannot be formed with sides 3, 4, and 8.

Using the Law of Cosines for SAS

Example 1 Solve ΔABC, given $\angle A = 50^\circ$, $b = 12$, and $c = 5$.

► *Solution*:



$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a^2 = 12^2 + 5^2 - 2 \cdot 5 \cdot 12\cos 50^\circ$$
$$a^2 = 91.865$$
$$a = 9.5846...$$

The Law of Sines may be used to find another angle of the triangle. To avoid obtaining two solutions for the angle, it is best to find the angle opposite the shortest side, since that angle is always acute.

$$\frac{\sin A}{a} = \frac{\sin C}{c} \rightarrow \frac{\sin 50^\circ}{9.58} = \frac{\sin C}{5} \rightarrow \sin C = \frac{5\sin 50^\circ}{9.58} \rightarrow \angle C = 23.6^\circ$$

Therefore $50^\circ + \angle B + 23.6^\circ = 180^\circ \rightarrow \angle B = 106.4^\circ$

Note: If $\angle B$ was solved for first, it would lead to a ***wrong*** *angle.*

$$\frac{\sin A}{a} = \frac{\sin B}{b} \rightarrow \frac{\sin 50^\circ}{9.58} = \frac{\sin B}{12} \rightarrow \sin B = \frac{12\sin 50^\circ}{9.58} \rightarrow \angle B = 73.6^\circ$$

The correct result for $\angle B$ is $180^\circ - 73.6^\circ = 106.4^\circ$. To avoid this problem, always find the smallest angle first in a SAS problem.

Summary of Law of Sines and Law of Cosines

Given	Method of Solving
ASA or AAS	1. Find the remaining angle using $\angle A + \angle B + \angle C = 180^\circ$. 2. Find the remaining sides using the Law of Sines.
ASS	Be aware of the ambiguous case. There may be two solutions. 1. Find an angle using the Law of Sines. 2. Find the remaining angle using $\angle A + \angle B + \angle C = 180^\circ$. 3. Find the remaining side using the Law of Sines.
SAS	1. Find the remaining side using the Law of Cosines. 2. Find the **smaller** of the two remaining angles using the Law of Sines. 3. Find the remaining angle using $\angle A + \angle B + \angle C = 180^\circ$.
SSS	1. Find the **largest** angle using the Law of Cosines. 2. Find the remaining angle by using the Law of Sines. 3. Find the remaining angle using $\angle A + \angle B + \angle C = 180^\circ$.