6.2 Trigonometric Function of Acute Angles

We have learned from studying trigonometry in grades 10 and 11 that for a fixed acute angle θ in a right triangle, the ratio of the length of the sides does not depend on the size of the triangle. The ratios depend on the measure of θ, therefore we can define trigonometric functions in terms of θ. Each ratio of a pair of lengths of sides of a right triangle is given a special name.

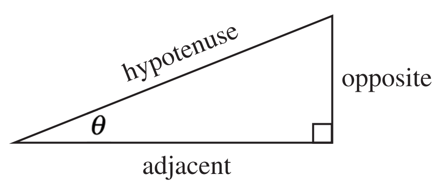
Trigonometric Functions of Acute Angles

For a given acute angle θ

Sine: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ Cosecant: $\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}}$

Cosine: $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ Secant: $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}}$

Tangent: $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$ Cotangent: $\csc\theta = \dfrac{\text{adjacent}}{\text{opposite}}$



Consider an angle θ in standard positon with $P(x, y)$ a point on the terminal side of θ. Then, by Pythagorean theorem, $x^2 + y^2 = r^2$ or $r = \sqrt{x^2 + y^2}$. The values of x, y and r determine the six trigonometric ratios for angle θ.

Trigonometry Ratios

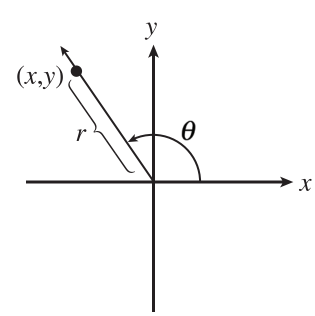
If θ is an angle in standard position with $P(x, y)$ a point on the terminal side of θ, then the six trigonometric ratios of angles θ are defined as follows:

$\sin\theta = \dfrac{y}{r}$ $\csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r}$ $\sec\theta = \dfrac{r}{x}$

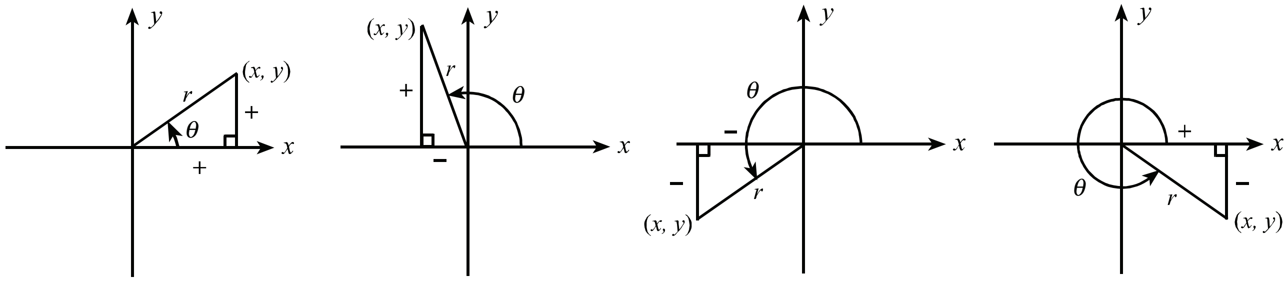
$\tan\theta = \dfrac{y}{x}$ $\csc\theta = \dfrac{x}{y}$

where $r = \sqrt{x^2 + y^2}$

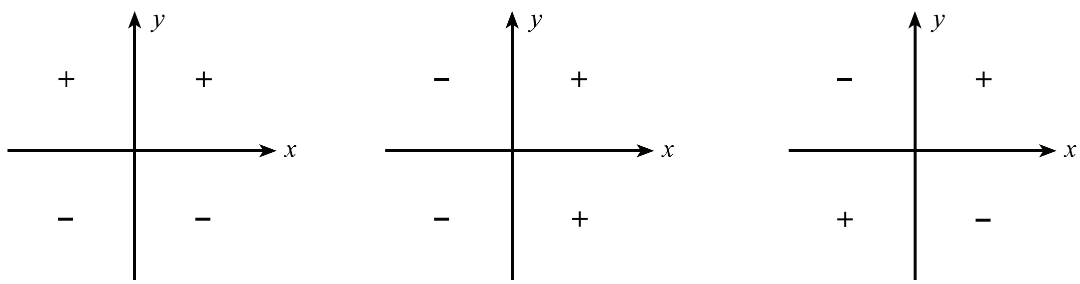


Algebraic Signs of the Trigonometric Functions

When selecting a point *P*(*x*, *y*) on the terminal side of angle θ, the quadrant in which θ is found will determine the algebraic sign of the trigonometric function. It will be either positive or negative.



Remember, $r=\sqrt{x^2+y^2}$ is always positive. Since $\sin\theta$ and $\csc\theta$ are always ratios of *y* and *r*, then $\sin\theta$ and $\csc\theta$ are positive where *y* is positive. Similarly, $\cos\theta$ and $\sec\theta$ are positive where *x* is positive. Also, since tan and cot are ratios of *x* and *y*, $\tan\theta$ and $\cot\theta$ are positive when *x* and *y* are both positive, or *x* and *y* are both negative, because a negative divided by a negative is positive.



Example 1 What quadrant has $\sin\theta<0$, $\tan\theta>0$?

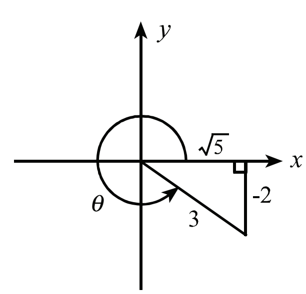
▶ *Solution:* $\sin\theta<0$ in quadrant III and IV, $\tan\theta>0$ in quadrant I and III, therefore, answer found in quadrant III.

Example 2 Determine $\cos\theta$ if $\csc\theta=-\frac{3}{2}$ and $\tan\theta<0$.

▶ *Solution:* $\csc\theta<0$ in quadrant III and IV, $\tan\theta<0$ in quadrant II and IV, therefore answer is found in quadrant IV. $\cos\theta$ is positive in quadrant IV.

$x^2+(-2)^2=3^2 \rightarrow x=\sqrt{5}$,

therefore $\cos\theta=\frac{\sqrt{5}}{3}$

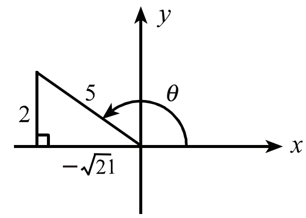


Example 3 Determine $\cot\theta$ if $\sin\theta = \frac{2}{5}$ and $\cos\theta < 0$.

▶ *Solution:* $\cos\theta < 0$ in quadrant II and III, $\sin\theta > 0$ in quadrant I and II, therefore answer found in quadrant II.

$x^2 + 2^2 = 5^2 \rightarrow x = -\sqrt{21}$ (x is negative in quadrant II)

therefore $\cot\theta = \frac{-\sqrt{21}}{2}$



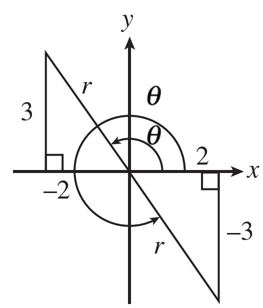
Example 4 Determine $\sec\theta$ if $\cot\theta = -\frac{2}{3}$.

▶ *Solution:* $\cot\theta < 0$ in quadrant II and IV

$r^2 = (-2)^2 + 3^2$

$r = \sqrt{13}$

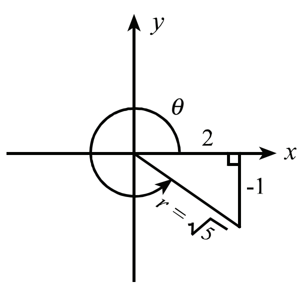
therefore, $\sec\theta = -\frac{\sqrt{13}}{2}$ or $\sec\theta = \frac{\sqrt{13}}{2}$



Example 5 Given the point (2, – 1) on the terminal side of angle θ, determine the value of all 6 trigonometric functions.

▶ *Solution:* $r^2 = (-1)^2 + 2^2$,

$r = \sqrt{5}$,

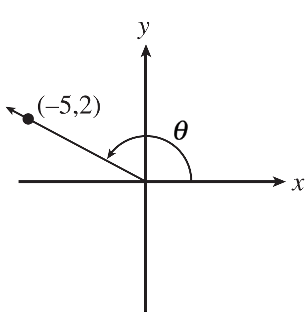


$\sin\theta = \frac{-1}{\sqrt{5}}$, $\cos\theta = \frac{2}{\sqrt{5}}$, $\tan\theta = \frac{-1}{2}$

$\csc\theta = \frac{\sqrt{5}}{-1} = -\sqrt{5}$, $\sec\theta = \frac{\sqrt{5}}{2}$, $\cot\theta = \frac{2}{-1} = -2$

Example 6 Determine $\sin\theta$ and $\cos\theta$ if θ is an angle in standard position whose terminal side is the graph $2x + 5y = 0,\ x \le 0$.

▶ *Solution:* Graph $2x + 5y = 0,\ x \le 0$



$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (-5)^2 + 2^2 \\ &= 29 \\ r &= \sqrt{29} \end{aligned},$$

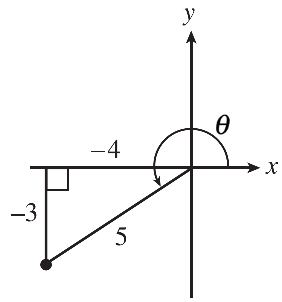
$$\sin\theta = \frac{2}{\sqrt{29}}$$

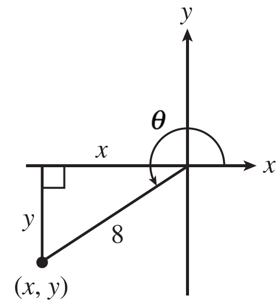
$$\cos\theta = \frac{-5}{\sqrt{29}}$$

Example 7 Determine the coordinate of the point 8 units from the origin in quadrant III and $\tan\theta = \frac{3}{4}$.

▶ *Solution:* Graph $\tan\theta = \frac{3}{4}$ in quadrant III. (Note: $x = -4, y = -3$)

$r = \sqrt{x^2 + y^2}$
$= \sqrt{3^2 + 4^2}$
$= 5$





by proportion

$\frac{-4}{5} = \frac{x}{8};\ x = \frac{-32}{5}$

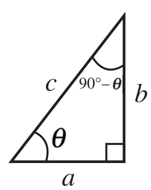
$\frac{-3}{5} = \frac{y}{8};\ y = \frac{-24}{5}$

Thus, coordinates $(x, y) = \left(-\frac{32}{5}, -\frac{24}{5}\right)$.

Example 8 If $\sin\theta = \frac{2}{3}$, find **a)** $\csc\theta$ and **b)** $\cos(90^\circ - \theta)$.

▶ *Solution:* **a)** $\sin\theta$ and $\csc\theta$ are reciprocals of each other, therefore if $\sin\theta = \frac{2}{3}$, $\csc\theta = \frac{3}{2}$.

b)



$\sin\theta = \frac{b}{c},\ \cos(90^\circ - \theta) = \frac{b}{c}$

Thus $\sin\theta = \cos(90^\circ - \theta)$,

So $\cos(90^\circ - \theta) = \frac{2}{3}$

Note: $\sin\theta = \cos(90^\circ - \theta),\ \cos\theta = \sin(90^\circ - \theta),\ \tan\theta = \cot(90^\circ - \theta)$
$\cot\theta = \tan(90^\circ - \theta),\ \sec\theta = \csc(90^\circ - \theta),\ \csc\theta = \sec(90^\circ - \theta)$
using the same reasoning as example 8b) above.

Example 9 Find all angles θ, $0^\circ \le \theta < 360^\circ$ such that $\sin\theta = -\cos\theta$.

▶ *Solution:* $\sin\theta > 0$ and $\cos\theta < 0$ in quadrant II with opposite = – adjacent

$\sin\theta < 0$ and $\cos\theta > 0$ in quadrant IV with adjacent = – opposite

The opposite and adjacent leg of the triangle must be the same length.

This must be a 45° reference angle in quadrants II and IV.

Thus, $\theta = 135^\circ$ and 315°.